But also

$$\frac{\Delta f}{\Delta x}=\frac{SD}{CD}<\frac{RD}{CD}=\frac{AD}{BD}=\frac{x+\Delta x}{f(x)+\Delta f}<\frac{x}{f(x)}+\frac{\Delta x}{f(x)}.$$

Now let $\Delta x \to 0^{+}$; we find

$$\frac{df}{dx}=\frac{x}{f(x)}.$$

(Although only the case that $\Delta x > 0$ has been studied here, it is easy to derive a similar equation for $\Delta x < 0$.)

Hence, f is a differentiable function of x. Solving the differential equation yields

$$f^{2}(x)=x^{2}+c,$$

where c is a constant. Obviously, if $x=0$ then $f(x)=b$. Hence $c=b^{2}$. The proof of the Pythagorean proposition is complete.

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Update on William Wernick's "Triangle Constructions with Three Located Points"

LEROY F. MEYERS*
The Ohio State University
Columbus, OH 43210

William Wernick's paper [9] contains a list of 139 problems, each of which asks for the Euclidean construction of a triangle from triples of "located points", i.e., points such as vertices, feet of altitudes, centroids, etc., whose location is given. Wernick was able to resolve nearly two-thirds of these problems, either by finding constructions or by proving redundancy of the data. (See figures and complete list of located points below.)

The notation below, introduced in [9], will be used in what follows for various points associated with a triangle. (See FIGURES 1–4.)

A, B, C, O	the three vertices, and circumcenter;
M_a, M_b, M_c, G	feet of the three medians, and centroid;
H_a, H_b, H_c, H	feet of the three altitudes, and orthocenter;
T_a, T_b, T_c, I	feet of the three internal angle bisectors, and incenter.

*We report with regret that Professor Meyers, the author of this article, died suddenly in November, 1995. Professor Meyers had a long-standing interest in geometry, problems, and undergraduate education. In the week he died, Professor Meyers was scheduled to speak—on topics related to this article—to a student mathematics club at Ohio State. During the period 1975–85, he served as Associate Editor of this MAGAZINE, including six years as Associate Problems Editor. —Ed.

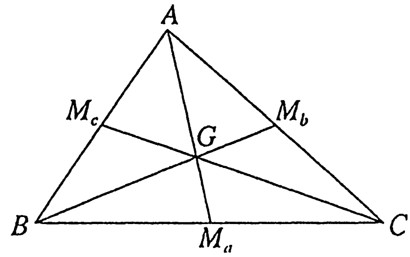

FIGURE 1

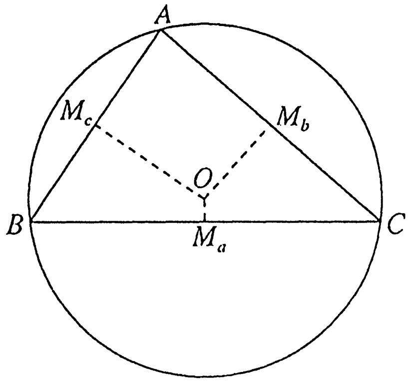

FIGURE 2

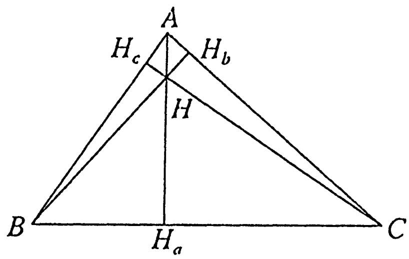

FIGURE 3

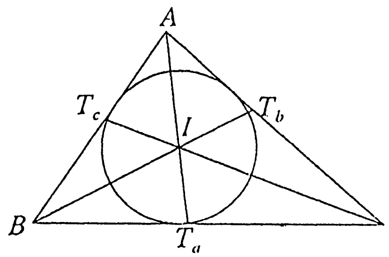

FIGURE 4

Since the appearance of Wernick's paper more than 13 years ago, about half of the problems left unresolved by him have been resolved, some positively but most negatively, and this paper is a report on the new results.

Table 1 contains a listing of these recently resolved problems, numbered as in [**9**], together with their resolutions. (A misprint in problem 102 is corrected.) Twenty problems remain unresolved.

TABLE 1 For each of the 30 triples of points listed, the problem of constructing the corresponding triangle ABC has been resolved by the author since the appearance of [**9**]. The triples are numbered as in that article. The letters **S**, **U**, and **L** designate that the problem of constructing a triangle from the given triple by Euclidean means is **S**olvable, **U**nsolvable, or **L**ocus-restricted, the last meaning that for a triangle to exist, one of the points must lie on a locus curve determined by the other two, but is not determined completely.

26. A, M_a, T_b	**U**	58. A, T_a, T_b	**S**	80. O, H, I	**U**	96. M_a, G, I	**S**	114. M_a, T_b, I	**U**
27. A, M_a, I	**S**	68. O, M_a, T_b	**U**	82. O, T_a, I	**S**	100. M_a, H_a, T_b	**U**	115. G, H_a, H_b	**U**
42. A, G, T_b	**U**	72. O, G, T_a	**U**	87. M_a, M_b, H	**S**	102. M_a, H_b, H_c	**L**	120. G, H, T_a	**U**
43. A, G, I	**S**	73. O, G, I	**U**	88. M_a, M_b, T_a	**U**	106. M_a, H_b, T_c	**U**	121. G, H, I	**U**
56. A, H, T_b	**U**	74. O, H_a, H_b	**U**	89. M_a, M_b, T_c	**U**	107. M_a, H_b, I	**U**	130. H_a, H, T_b	**U**
57. A, H, I	**S**	79. O, H, T_a	**U**	95. M_a, G, T_b	**U**	108. M_a, H, T_a	**U**	131. H_a, H, I	**S**

It is an interesting challenge to verify the results shown in the table. One sample verification is given below; the remaining verifications (and extensions!) are left to the interested reader, who may obtain further information from the author.

Algebra, often in connection with analytic geometry, can be used to prove that there is no Euclidean construction from certain triples of located points. All such proofs proceed by contradiction, and depend on Gauss's criterion for Euclidean constructibility. The following corollary of Gauss's theorem, quoted from [**4**, p. 33], is useful (see also [**3**, p. 550]).

THEOREM 1. *It is impossible to construct with ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root, when the unit of length is given.*

Problem 115. Given G, H_a, H_b.

Particular positions are chosen for the located points, and it is shown that they determine a triangle for which there is no straightedge and compasses construction. Let the located points in a rectangular coordinate system be $G = (1, \frac{2}{3})$, $H_a = (0, 1)$, and $H_b = (0, -1)$. (See FIGURE 5.)

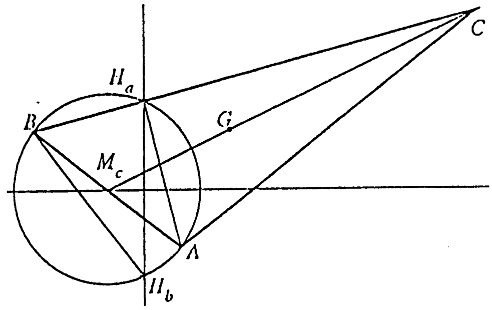

FIGURE 5

Since $\angle AH_aB = \angle AH_bB = 90°$, the points H_a and H_b lie on the circle having the segment AB as diameter and M_c as center. Then M_c lies on the perpendicular bisector of the segment H_aH_b. Hence $M_c = (x, 0)$ for some real number x. Since G is 2/3 of the way from C to M_c, we have $C = (3 - 2x, 2)$. Suppose that $A = (u, v)$ for some real numbers u and v. Then $B = 2M_c - A = (2x - u, -v)$. Since C, A, and H_b are collinear, the slopes of the lines AH_b and H_bC must be equal. Thus

$$\frac{v+1}{u} = \frac{3}{3-2x}.$$

Similarly, collinearity of C, B, and H_a yields

$$\frac{v+1}{u-2x} = \frac{1}{3-2x}.$$

If we divide the first equation by the second and solve for u, we obtain

$$u = -x \quad \text{and then} \quad v = \frac{x+3}{2x-3}.$$

Since M_c is the circumcenter of right triangle ABH_b, we have

$$x^2 + 1 = (x - u)^2 + v^2,$$

and so substitution with simplification yields

$$2x^3 - 6x^2 + 4x + 3 = 0,$$

which has no rational root. Hence by the theorem quoted above, x is nonconstructible. There *is* a triangle having the given located points, with $x \approx -0.4311$ and the nonconstructible vertices $A \approx (0.4311, -0.6651)$, $B \approx (-1.2934, 0.6651)$, and $C \approx (3.8623, 2)$.

Readers are invited to fill in the blanks still remaining in Wernick's problem list. The following 20 problems are open:

77. O, H_a, T_b	109. M_a, H, T_b	118. G, H_a, T_b	127. H_a, H_b, T_c	135. H_a, T_b, I
78. O, H_a, I	110. M_a, H, I	119. G, H_a, I	128. H_a, H_b, I	136. H, T_a, T_b
81. O, T_a, T_b	111. M_a, T_a, T_b	122. G, T_a, T_b	132. H_a, T_a, T_b	137. H, T_a, I
90. M_a, M_b, I	113. M_a, T_b, T_c	123. G, T_a, I	134. H_a, T_b, T_c	138. T_a, T_b, T_c

Note. Not much seems to have been published on the construction of triangles using located points. Most works on geometric constructions, such as the excellent [7], treat triangle construction problems only from the point of view of "parts", such as sides, angles, medians, altitudes, and the like. A systematic list of "parts" problems, together with solutions to some of them, can be found in [8], and a smaller systematic list, together with solutions, is [5]; the corresponding unsolvable "parts" problems are treated in [6]. Recent "challenge" columns are [1, 2].

Acknowledgment. The author thanks William Wernick for his encouragement during the work leading to this paper.

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Ceva's and Menelaus' Theorems and Their Converses via Centroids

MURRAY S. KLAMKIN
University of Alberta
Edmonton, Alberta, Canada T6G 2G1

SIDNEY H. KUNG
Jacksonville University
Jacksonville, FL 32211

It appears that present-day students do not know much about applications of centroids to geometry. Perhaps this note may help rectify this situation. For further applications, see [1], [2], and [4].

Ceva's theorem states that if AD, BE, and CF are three concurrent cevians of a triangle ABC as in FIGURE 1, then

$$\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)=1. \quad (1)$$