This coordinate-free method for finding the area under an involute was used in several solutions listed in the bibliography. It is presently a rather neglected topic of undergraduate teaching, but arises naturally in this setting if you wish to revive it.

We now have the total area the goat grazes:

$$\text{area grazed} = 2 \text{ bad areas} + \text{area of semicircle}$$
$$= \frac{a^2k^2}{6}(2k+3\pi).$$

We can set up and solve the required equation:

$$\text{area grazed} = \tfrac{1}{2}\text{ area field}$$
$$\frac{a^2k^2}{6}(2k+3\pi) = \tfrac{1}{2}\pi a^2$$
$$2k^3+3\pi k^2 = 3\pi.$$

At least this goat provides us with a polynomial equation, and a cubic at that. I found it interesting to find the exact solution. We eliminate the quadratic term with the substitution $k = x - \pi/2$. After a little simplification we obtain

$$4x^3 - 3\pi^2 x = 6\pi - \pi^3.$$

Now we can make the substitution $x = \pi\cos\phi$, a nice trick, which yields

$$\cos 3\phi = \frac{6}{\pi^2} - 1$$
$$\phi = \frac{1}{3}\arccos\left(\frac{6}{\pi^2} - 1\right).$$

Tracing back our steps, we obtain

$$k = \pi\cos\phi - \frac{\pi}{2},$$

with the value of ϕ given above. An approximate solution is

$$k \approx 0.9150896408.$$

In this variation the rope should be slightly shorter than the radius of the field—about 0.915 of the radius.

My experience with these two goat problems was to approach them using a "standard," "tried-and-true" method by introducing coordinates. In each case I later discovered a simpler, more satisfying solution, thus increasing my appreciation of the problems.

I find both of these problems interesting. Each is easy to comprehend. Each involves some difficulties, but the difficulties are surmountable using some fine elementary mathematics. Also, the problems invite further questions. I'm presently looking into an n-dimensional generalization of the grazing goat which I hope to write up soon.

I wish to thank Cathi Colin, Fred Safier, and Peter Renz for discussions and suggestions concerning grazing goats, Howard Eves for his informative letter, and a referee for many suggestions.

References

Here are some grazing goat problems from early issues of *The American Mathematical Monthly*. The animal, when there is one, is a horse. A mathematician who has herded goats tells me that they are much too independent to submit to tethering.

[1] Geometry Problem 30; proposed in 1 (1894) 132; solution in 1 (1894) 395–96.

[2] Arithmetic Problem 32; proposed in 1 (1894) 266; solution in 1 (1894) 431.
Horse tethered to a corner of a square building.

[3] Geometry Problem 34; proposed as Problem 33 in 1 (1894) 317; solution as Problem 34 in 2 (1895) 48-49.
Both problems—a horse can graze inside the fence and then outside the fence. The solution published for outside the fence seems obviously incorrect.

[4] Calculus Problem 37; proposed in 2 (1895) 52; solution in 2 (1895) 277–78.
Two mules graze together, but on opposite sides of the fence.

[5] Calculus Problem 55; proposed in 3 (1896) 148; solutions in 4 (1897) 17–18.
Both problems again—a horse can graze outside the fence and then inside the fence. Involutes enter into the solutions.

[6] Calculus Problem 69; proposed in 5 (1898) 29; solution in 5 (1898) 111; note in 5 (1898) 177.
Horse is tethered to a sliding ring outside an elliptical field, perhaps the most exotic problem. The note points out that the published solution is incorrect, but a correct solution is not offered.

[7] Arithmetic Problem 93; proposed as Problem 91 in 5 (1898) 60; solution as Problem 93 in 5 (1898) 170–71.
Another horse is tethered to the corner of a barn. But six different answers were received.

[8] Geometry Problem 103; proposed in 5 (1898) 217; solutions in 5 (1898) 295–96.
A horse is grazing on the edge of a circular pond. Three solutions reflect different assumptions about whether the rope can stretch across the pond. Curiously, the answer is almost the same in each case.

[9] Calculus Problem 103; proposed in 6 (1899) 289; remark in 7 (1900) 267–68.
Essentially the same as Calculus Problem 69—a horse is grazing outside an elliptical field tethered to a sliding ring. No solution was received and, so far as I know, these two problems remain unsolved.

Grazing goat problems and some variants from recent sources.

[10] V. W. B., An iterative process: the goat's share revisited, Math. Gaz., 65 (1981) 137–39.
The classical goat; discussion of iterative calculator solution of the formidable-looking arcsine equation.

[11] Howard P. Dinesman, Superior Mathematical Puzzles, Simon and Schuster, New York, 1968; Puzzle 8 on p. 20 and Puzzle 53 on p. 71.
Puzzle 8 is very easy; in Puzzle 53 a goat is tethered to a circular silo.

[12] Jordi Dou, Solution to Problem S19 (proposed by Harley Flanders), Amer. Math. Monthly, 88 (1981) 147.

[13] Henry Earnest Dudeney, Amusements in Mathematics, Dover, New York, 1958 (reprint of 1917 edition), Problem 196 on p. 53.
Goat grazing in an equilateral triangle; easy.

[14] L. A. Graham, Ingenious Mathematical Problems, Dover, New York, 1959, p. 6.
This has a literary twist, and is called "Mrs. Miniver's Problem."

[15] S. I. Jones, Mathematical Nuts, S. I. Jones Co., 1932.
Note the quick calculation of the area of the involute.

[16] P. M. H. Kendall and G. M. Thomas, Mathematical Puzzles for the Connoisseur, Thomas Y. Crowell, New York, 1962, pp. 24–25.
A macabre goat is tethered to a mausoleum in a circular field. A nice involute problem.

[17] L. H. Longley-Cook, Work This One Out, Fawcett, 1960, Problem 69.

[18] James E. Schultz and Bert K. Waits, A new look at some old problems in light of the hand calculator, TYCMJ, 10 (1979) 20–27.
Discusses calculator solutions of three classical problems—one of which is the grazing goat. A simple derivation using sectors of a complicated-looking equation.

Triangle Constructions with Three Located Points

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Given a triangle ABC we can construct its medians, AM_a, BM_b, and CM_c which are concurrent at the centroid G (Figure 1). Now suppose we erase almost all of this figure, leaving only the points A, B, and M_a *in position*. Can we reconstruct the original figure? Yes, very easily, since if we double the segment BM_a we get the point C.

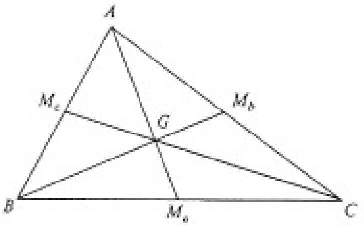

FIGURE 1

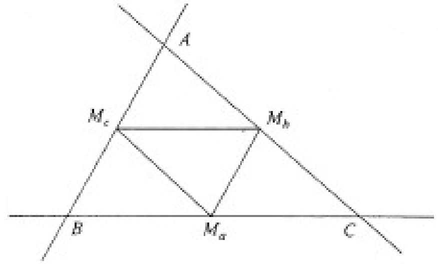

FIGURE 2

Suppose we erase all but the three midpoints M_a, M_b, M_c, *in location*; can we again reconstruct the original figure? One solution is to start by drawing a line through M_c parallel to the line M_aM_b, then another line through M_a parallel to the line M_bM_c. These two constructed lines will intersect in point B, a vertex of the original triangle, and the rest of the construction follows easily (FIGURE 2). We indicate this solution as follows:

$$//(M_c, M_aM_b) \cap //(M_a, M_bM_c) = B; \qquad 2(BM_c) = A, \qquad 2(BM_a) = C.$$

It is interesting to investigate the construction of a triangle ABC, given certain triples of located points selected from the following list of sixteen points (see FIGURES 1, 3, 4, 5):

A, B, C, O	Three vertices and circumcenter
M_a, M_b, M_c, G	Three feet of the medians, and centroid
H_a, H_b, H_c, H	Three feet of the altitudes, and orthocenter
T_a, T_b, T_c, I	Three feet of the internal angle bisectors, and incenter.

In these problems we may take two approaches: (1) we assume that a triangle has been erased, except for three located points, and we try to recover that original triangle; or (2) we choose any three distinct points of the plane and designate these as three particular points among the list of sixteen, then try to construct a triangle to fit. It is clear that the second approach includes the first and is a little more general, raising questions of constructibility and redundancy. Since it is more interesting, it is the approach we shall use.

The list in TABLE 1 is a careful compilation of exactly 139 such problems, all significantly distinct. For example, the selection of the triple of points to be two vertices and the centroid of a triangle could be listed as A, B, G; or B, C, G; or A, C, G, which are surely not significantly distinct. Only the first choice is listed (problem 4). Some selections of triples are redundant, that is, one of the points can be determined from the other two. For example, the triple A, B, M_c is redundant since M_c is the midpoint of segment AB (clearly any two of these three points determine the third). Such redundant triples in TABLE 1 are noted with the letter **R**; I have found three such triples in this list.

Note also that some selections, while not redundant, are still restricted as to choice of the points; the triple A, B, O is such a selection, since the circumcenter O must lie on the perpendicular bisector of the side AB. If the point designated O does lie on the perpendicular bisector of the segment AB, then the third vertex C of the triangle may lie anywhere on the circumcircle with center O and radius OA. If the point designated as O does not lie on the perpendicular bisector of segment AB, then no solution is possible. In such a situation, the locus restriction gives us either infinitely many or no solutions to the problem. Twenty-two of the problems in TABLE 1 have been identified as being of this type; they are designated with the symbol **L**.

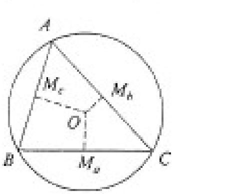

FIGURE 3

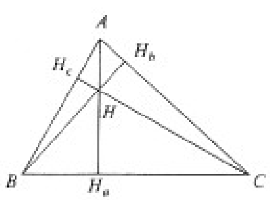

FIGURE 4

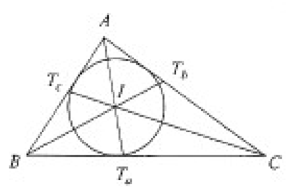

FIGURE 5

Of the remaining 114 problems, I have found solutions for only 65 of these, which are indicated on the list with the symbol **S**. Many of these 65 are quite easy and straightforward to solve. Some are harder and would challenge any student of geometry. Problems 69, 76, and 101 in particular are real challenges; they appear as Proposal 1149 in the Problems section of this issue of this *Magazine*, p. 236.

Fortunately, some problems are related, in the sense that if you solve one of them, you have ready solutions to some others. Consider, for example, problems 26, 42, and 95, whose triples of given points are as follows:

$$26:\quad A, M_a, T_b$$
$$42:\quad A, G, T_b$$
$$95:\quad M_a, G, T_b.$$

Since the triple A, M_a, G is redundant (problem 21), it follows that a solution to any one of the problems 26, 42, 95 leads to immediate solutions of the other two problems. Other sets of such related problems are 27, 43, 96; 72, 79, 120; and 73, 80, 121.

Our list in TABLE 1 contains 41 seemingly independent, presently unsolved problems. It may be that some of these lead to provably impossible constructions, in which case such a proof would

1. A, B, O	L	29. A, M_b, G	S	57. A, H, I		85. M_a, M_b, H_a	S	113. M_a, T_b, T_c	
2. A, B, M_a	S	30. A, M_b, H_a	L	58. A, T_a, T_b		86. M_a, M_b, H_c	S	114. M_a, T_b, I	
3. A, B, M_c	R	31. A, M_b, H_b	L	59. A, T_a, I	L	87. M_a, M_b, H		115. G, H_a, H_b	
4. A, B, G	S	32. A, M_b, H_c	L	60. A, T_b, T_c	S	88. M_a, M_b, T_a		116. G, H_a, H	S
5. A, B, H_a	L	33. A, M_b, H	S	61. A, T_b, I	S	89. M_a, M_b, T_c		117. G, H_a, T_a	S
6. A, B, H_c	L	34. A, M_b, T_a	S	62. O, M_a, M_b	S	90. M_a, M_b, I		118. G, H_a, T_b	
7. A, B, H	S	35. A, M_b, T_b	L	63. O, M_a, G	S	91. M_a, G, H_a	L	119. G, H_a, I	
8. A, B, T_a	S	36. A, M_b, T_c	S	64. O, M_a, H_a	L	92. M_a, G, H_b	S	120. G, H, T_a	
9. A, B, T_c	L	37. A, M_b, I	S	65. O, M_a, H_b	S	93. M_a, G, H	S	121. G, H, I	
10. A, B, I	S	38. A, G, H_a	L	66. O, M_a, H	S	94. M_a, G, T_a	S	122. G, T_a, T_b	
11. A, O, M_a	S	39. A, G, H_b	S	67. O, M_a, T_a	L	95. M_a, G, T_b		123. G, T_a, I	
12. A, O, M_b	L	40. A, G, H	S	68. O, M_a, T_b		96. M_a, G, I		124. H_a, H_b, H_c	S
13. A, O, G	S	41. A, G, T_a	S	69. O, M_a, I	S	97. M_a, H_a, H_b	S	125. H_a, H_b, H	S
14. A, O, H_a	S	42. A, G, T_b		70. O, G, H_a	S	98. M_a, H_a, H	L	126. H_a, H_b, T_a	S
15. A, O, H_b	S	43. A, G, I		71. O, G, H	R	99. M_a, H_a, T_a	L	127. H_a, H_b, T_c	
16. A, O, H	S	44. A, H_a, H_b	S	72. O, G, T_a		100. M_a, H_a, T_b		128. H_a, H_b, I	
17. A, O, T_a	S	45. A, H_a, H	L	73. O, G, I		101. M_a, H_a, I	S	129. H_a, H, T_a	L
18. A, O, T_b	S	46. A, H_a, T_a	L	74. O, H_a, H_b		102. M_a, H_b, H_c	S	130. H_a, H, T_b	
19. A, O, I	S	47. A, H_a, T_b	S	75. O, H_a, H	S	103. M_a, H_b, H	S	131. H_a, H, I	
20. A, M_a, M_b	S	48. A, H_a, I	S	76. O, H_a, T_a	S	104. M_a, H_b, T_a	S	132. H_a, T_a, T_b	
21. A, M_a, G	R	49. A, H_b, H_c	S	77. O, H_a, T_b		105. M_a, H_b, T_b	S	133. H_a, T_a, I	S
22. A, M_a, H_a	L	50. A, H_b, H	L	78. O, H_a, I		106. M_a, H_b, T_c		134. H_a, T_b, T_c	
23. A, M_a, H_b	S	51. A, H_b, T_a	S	79. O, H, T_a		107. M_a, H_b, I		135. H_a, T_b, I	
24. A, M_a, H	S	52. A, H_b, T_b	L	80. O, H, I		108. M_a, H, T_a		136. H, T_a, T_b	
25. A, M_a, T_a	S	53. A, H_b, T_c	S	81. O, T_a, T_b		109. M_a, H, T_b		137. H, T_a, I	
26. A, M_a, T_b		54. A, H_b, I	S	82. O, T_a, I		110. M_a, H, I		138. T_a, T_b, T_c	
27. A, M_a, I		55. A, H, T_a	S	83. M_a, M_b, M_c	S	111. M_a, T_a, T_b		139. T_a, T_b, I	S
28. A, M_b, M_c	S	56. A, H, T_b		84. M_a, M_b, G	S	112. M_a, T_a, I	S		

TABLE 1. **For each of the 139 triples of points listed, construct the corresponding triangle ABC. Problems solved by the author are noted with an S, L, or R to designate a solution triangle ABC, a locus-dependent solution, or a redundant triple, respectively.**

constitute a solution. However, I have the firm conviction that they can all be "done," and that eventually every one of the 139 problems will have the appropriate **R** or **L** or **S** designation. I welcome any companionship and friendly competition on the last 41 (which incidentally is a prime number with pleasant associations).

I close with a few solutions to whet your appetite.

PROBLEM 18. Given points A, O, T_b.

Solution. $\perp(O, AT_b) \cap \mathrm{Cir}(O, OA) = K$; $KT_b \cap \mathrm{Cir}(O, OA) = B$; $AT_b \cap \mathrm{Cir}(O, OA) = C$. In words, the perpendicular from O to the line AT_b will meet the circle with center O and radius OA at the point K; the line KT_b will meet the circle with center O and radius OA at the point B; (now it's your turn) $AT_b \cap \mathrm{Cir}(O, OA) = C$.

PROBLEM 49. Given points A, H_b, H_c.

Solution. $\mathrm{Cir}(A, H_b, H_c) = \mathrm{Cir}(K)$; $\mathrm{Diam\,Cir}(K) = AH$; $H_bH \cap AH_c = B$; $H_cH \cap AH_b = C$. That is, the circle which goes through points A, H_b, and H_c (the circumcircle of triangle AH_bH_c) we call circle K; the diameter of circle K which has one end at A has the other end at the point H; the line H_bH will meet the line AH_c at the point B; and now $H_cH \cap AH_b = C$.

PROBLEM 65. Given points O, M_a, H_b.

Solution. $\perp(M_a, M_aO) = xM_ay$; $\mathrm{Cir}(M_a, M_aH_b) \cap xM_ay = B, C$; $\mathrm{Cir}(O, OB) = \mathrm{Cir}(K)$; $\mathrm{Cir}(K) \cap CH_b = A$. The perpendicular at the point M_a to the line M_aO we call the line xM_ay; the circle with center M_a and radius M_aH_b will meet the line xM_ay at points B and C; the circle with center O and radius OB is called circle K; circle K will meet the line CH_b at point A.

PROBLEM 103. Given points M_a, H_b, H.

Solution. $\perp(M_a, H_bH) \cap H_bH = Q$; $2(H_bQ) = B$; $2(BM_a) = C$; $\perp(B, CH) \cap CH_b = A$. The perpendicular from M_a to the line H_bH will meet the line H_bH at the point Q (that is, Q is the orthogonal projection of M_a to line H_bH); extend segment H_bQ to double its length to get point B; double the segment BM_a to get point C; the perpendicular from point B to line CH will meet line CH_b at point A.

A Papal Conclave: Testing the Plausibility of a Historical Account

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Among the stories to be found in Valérie Pirie's *The Triple Crown*, a gossipy and unreliable account of the Papal conclaves since 1458, is one having to do with the election of 1513 that appears particularly suspicious to a reader with some mathematical abilities.

> Twenty-five cardinals entered the conclave. The absence of the French element left practically only two contending parties—the young and the old. The former had secretly settled on Giovanni de' Medici; the second openly supported S. Giorgio, England's candidate.... The Sacred College had been assembled almost a week before the first serious scrutiny took place. Many of the cardinals, wishing to temporise and conceal their real intentions, had voted for the man they considered least likely to have any supporters. As luck would have it, thirteen prelates had selected the same outsider, with the result that they all but elected Arborense, the most worthless nonentity present. This narrow shave gave the Sacred College such a shock that its members determined to come to some agreement which would put matters on a more satisfactory basis for both parties.
>
> [5, p. 49]

We may assume that all but a handful of the twenty-five cardinals were recognized as having no chance whatsoever of winning seventeen votes, the two-thirds majority required for an election. Of these, a smaller number m would have been considered to have absolutely no supporters; this number cannot have been too small, or there would not have been so great a surprise when one man among these "least likely to have any supporters" won thirteen votes. If we further assume that r cardinals, $17 \leqslant r \leqslant 25$, participated in the strategy, and that each voted at random on that first ballot for one of the m "dark horses," *what was the probability that one of the m would receive thirteen votes*? More importantly, since their eminences were terrified when this event actually occurred and determined to change their ways, we may wonder what was the probability of anyone actually being elected by this method. We shall presume that the $25 - r$ prelates not participating in the maneuver each voted for one of the serious candidates.

It must be pointed out that it is quite possible that the m cardinals who were considered to have no supporters were ranked according to undesirability, in which case the voting would not have been at random. In our analysis which follows we assume that the m undesirable candidates were deemed "equally undesirable."

The number of different ways in which r votes can be randomly cast among m candidates is the same as the number of different ways r indistinguishable objects can be distributed among m cells, that is, $\binom{m+r-1}{r}$ (see [2, p. 31], and take $n = m - 1$). The number of different ways in which a particular one of the m candidates can get thirteen votes is the same as the number of ways in which the remaining $r - 13$ votes can be divided among the remaining $m - 1$ candidates, i.e., $\binom{m+r-15}{r-13}$. The probability of the event described by Pirie is therefore

$$\frac{m\binom{m+r-15}{r-13}}{\binom{m+r-1}{r}}, \qquad m > 1.$$

These probabilities are listed in TABLE 1. In compiling this table we have adopted the praiseworthy axiom of Emile Borel, presented in his discussion of the Petersburg Paradox in [1], that, as a working rule, events of probability less than .001 should be considered impossible. The table displays only those columns for which $17 \leqslant r \leqslant 20$, since a small number of pious cardinals would

m \ r	17	18	19	20
1	0	0	0	0
2	.1111	.1053	.1000	.0952
3	.0877	.0948	.1000	.1039
4	.0526	.0632	.0727	.0813
5	.0292	.0383	.0474	.0565
6	.0159	.0225	.0296	.0373
7	.0087	.0131	.0182	.0241
8	.0048	.0077	.0112	.0155
9	.0027	.0046	.0069	.0100
10	.0016	.0028	.0043	.0065
11	0	.0017	.0027	.0042
12	0	0	.0017	.0027
13	0	0	.0011	.0018
14	0	0	0	.0012
15	0	0	0	0

TABLE 1. **Probability of one of the "least likely" candidates receiving exactly 13 votes.**

But also

$$\frac{\Delta f}{\Delta x} = \frac{SD}{CD} < \frac{RD}{CD} = \frac{AD}{BD} = \frac{x + \Delta x}{f(x) + \Delta f} < \frac{x}{f(x)} + \frac{\Delta x}{f(x)}.$$

Now let $\Delta x \to 0^+$; we find

$$\frac{df}{dx} = \frac{x}{f(x)}.$$

(Although only the case that $\Delta x > 0$ has been studied here, it is easy to derive a similar equation for $\Delta x < 0$.)

Hence, f is a differentiable function of x. Solving the differential equation yields

$$f^2(x) = x^2 + c,$$

where c is a constant. Obviously, if $x = 0$ then $f(x) = b$. Hence $c = b^2$. The proof of the Pythagorean proposition is complete.

REFERENCE

1. Elisha Scott Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, Washington, DC, 1968, pp. 244–245.

Update on William Wernick's "Triangle Constructions with Three Located Points"

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William Wernick's paper [9] contains a list of 139 problems, each of which asks for the Euclidean construction of a triangle from triples of "located points", i.e., points such as vertices, feet of altitudes, centroids, etc., whose location is given. Wernick was able to resolve nearly two-thirds of these problems, either by finding constructions or by proving redundancy of the data. (See figures and complete list of located points below.)

The notation below, introduced in [9], will be used in what follows for various points associated with a triangle. (See FIGURES 1–4.)

A, B, C, O	the three vertices, and circumcenter;
M_a, M_b, M_c, G	feet of the three medians, and centroid;
H_a, H_b, H_c, H	feet of the three altitudes, and orthocenter;
T_a, T_b, T_c, I	feet of the three internal angle bisectors, and incenter.

*We report with regret that Professor Meyers, the author of this article, died suddenly in November, 1995. Professor Meyers had a long-standing interest in geometry, problems, and undergraduate education. In the week he died, Professor Meyers was scheduled to speak—on topics related to this article—to a student mathematics club at Ohio State. During the period 1975–85, he served as Associate Editor of this MAGAZINE, including six years as Associate Problems Editor. —Ed.

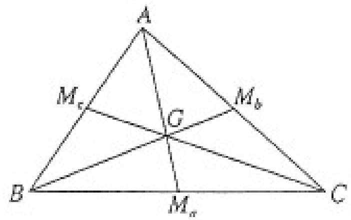

FIGURE 1

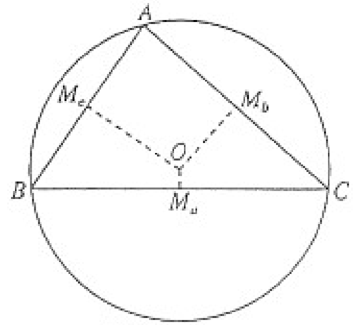

FIGURE 2

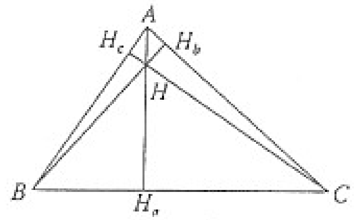

FIGURE 3

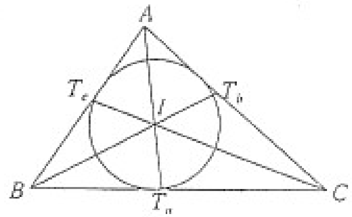

FIGURE 4

Since the appearance of Wernick's paper more than 13 years ago, about half of the problems left unresolved by him have been resolved, some positively but most negatively, and this paper is a report on the new results.

Table 1 contains a listing of these recently resolved problems, numbered as in [9], together with their resolutions. (A misprint in problem 102 is corrected.) Twenty problems remain unresolved.

TABLE 1 For each of the 30 triples of points listed, the problem of constructing the corresponding triangle ABC has been resolved by the author since the appearance of [9]. The triples are numbered as in that article. The letters **S**, **U**, and **L** designate that the problem of constructing a triangle from the given triple by Euclidean means is **S**olvable, **U**nsolvable, or **L**ocus-restricted, the last meaning that for a triangle to exist, one of the points must lie on a locus curve determined by the other two, but is not determined completely.

26.	A, M_a, T_b	**U**	58.	A, T_a, T_b	**S**	80.	O, H, I	**U**	96.	M_a, G, I	**S**	114.	M_a, T_b, I	**U**
27.	A, M_a, I	**S**	68.	O, M_a, T_b	**U**	82.	O, T_a, I	**S**	100.	M_a, H_a, T_b	**U**	115.	G, H_a, H_b	**U**
42.	A, G, T_b	**U**	72.	O, G, T_a	**U**	87.	M_a, M_b, H	**S**	102.	M_a, H_b, H_c	**L**	120.	G, H, T_a	**U**
43.	A, G, I	**S**	73.	O, G, I	**U**	88.	M_a, M_b, T_a	**U**	106.	M_a, H_b, T_c	**U**	121.	G, H, I	**U**
56.	A, H, T_b	**U**	74.	O, H_a, H_b	**U**	89.	M_a, M_b, T_c	**U**	107.	M_a, H_b, I	**U**	130.	H_a, H, T_b	**U**
57.	A, H, I	**S**	79.	O, H, T_a	**U**	95.	M_a, G, T_b	**U**	108.	M_a, H, T_a	**U**	131.	H_a, H, I	**S**

It is an interesting challenge to verify the results shown in the table. One sample verification is given below; the remaining verifications (and extensions!) are left to the interested reader, who may obtain further information from the author.

Algebra, often in connection with analytic geometry, can be used to prove that there is no Euclidean construction from certain triples of located points. All such proofs proceed by contradiction, and depend on Gauss's criterion for Euclidean constructibility. The following corollary of Gauss's theorem, quoted from [4, p. 33], is useful (see also [3, p. 550]).

THEOREM 1. *It is impossible to construct with ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root, when the unit of length is given.*

Problem 115. Given G, H_a, H_b.

Particular positions are chosen for the located points, and it is shown that they determine a triangle for which there is no straightedge and compasses construction. Let the located points in a rectangular coordinate system be $G=(1,\frac{2}{3})$, $H_a=(0,1)$, and $H_b=(0,-1)$. (See FIGURE 5.)

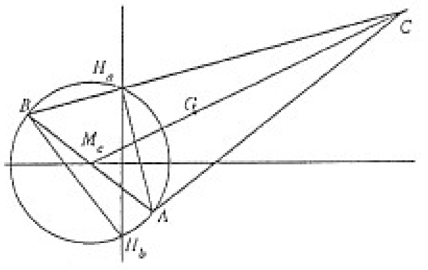

FIGURE 5

Since $\angle AH_aB = \angle AH_bB = 90°$, the points H_a and H_b lie on the circle having the segment AB as diameter and M_c as center. Then M_c lies on the perpendicular bisector of the segment H_aH_b. Hence $M_c=(x,0)$ for some real number x. Since G is $2/3$ of the way from C to M_c, we have $C=(3-2x,2)$. Suppose that $A=(u,v)$ for some real numbers u and v. Then $B=2M_c-A=(2x-u,-v)$. Since C, A, and H_b are collinear, the slopes of the lines AH_b and H_bC must be equal. Thus

$$\frac{v+1}{u}=\frac{3}{3-2x}.$$

Similarly, collinearity of C, B, and H_a yields

$$\frac{v+1}{u-2x}=\frac{1}{3-2x}.$$

If we divide the first equation by the second and solve for u, we obtain

$$u=-x \quad\text{and then}\quad v=\frac{x+3}{2x-3}.$$

Since M_c is the circumcenter of right triangle ABH_b, we have

$$x^2+1=(x-u)^2+v^2,$$

and so substitution with simplification yields

$$2x^3-6x^2+4x+3=0,$$

which has no rational root. Hence by the theorem quoted above, x is nonconstructible. There *is* a triangle having the given located points, with $x\approx-0.4311$ and the nonconstructible vertices $A=(0.4311,-0.6651)$, $B=(-1.2934,0.6651)$, and $C\approx(3.8623,2)$.

Readers are invited to fill in the blanks still remaining in Wernick's problem list. The following 20 problems are open:

77. O, H_a, T_b	109. M_a, H, T_b	118. G, H_a, T_b	127. H_a, H_b, T_c	135. H_a, T_b, I
78. O, H_a, I	110. M_a, H, I	119. G, H_a, I	128. H_a, H_b, I	136. H, T_a, T_b
81. O, T_a, T_b	111. M_a, T_a, T_b	122. G, T_a, T_b	132. H_a, T_a, T_b	137. H, T_a, I
90. M_a, M_b, I	113. M_a, T_b, T_c	123. G, T_a, I	134. H_a, T_b, T_c	138. T_a, T_b, T_c

Note. Not much seems to have been published on the construction of triangles using located points. Most works on geometric constructions, such as the excellent [7], treat triangle construction problems only from the point of view of "parts", such as sides, angles, medians, altitudes, and the like. A systematic list of "parts" problems, together with solutions to some of them, can be found in [8], and a smaller systematic list, together with solutions, is [5]; the corresponding unsolvable "parts" problems are treated in [6]. Recent "challenge" columns are [1, 2].

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Ceva's and Menelaus' Theorems and Their Converses via Centroids

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It appears that present-day students do not know much about applications of centroids to geometry. Perhaps this note may help rectify this situation. For further applications, see [1], [2], and [4].

Ceva's theorem states that if AD, BE, and CF are three concurrent cevians of a triangle ABC as in FIGURE 1, then

$$\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)=1. \tag{1}$$